Dynamic Asset Allocation and Consumption Growth: A Specific Case with an Example

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This study derives a Keynes-Ramsey rule (KRR) for a simple optimal consumption-investment problem with constant investment opportunities by using the stochastic control framework for infinite and finite models. The deterministic part of the KRR is typically the average consumption growth rate, containing two constants and one time-varying term. It contains (1) the usual difference between the interest rate and time preference rate, (2) the term capturing the impact of market prices of risk, and (3) the time horizon related term which is the effect of introducing time horizon into the problem. The first two terms are also the KRR for the infinite time horizon model whereas the addition of the last term creates the KRR for the finite model. The study finds that interest rate movements not only positively affect consumption growth, but also indirectly and negatively affect the growth via the market price of risks. A numerical example finds that (1) the existence of the time-varying term is great when consumption-investment time horizon is short and investors are more afraid of risks; and (2) interest rates have a negative relationship with both infinite and finite consumption growth and the effect is especially substantial when investors are less risk-averse. The second result is against previous theoretical and empirical studies.

JEL Codes: C61, D92, and G11

1. Introduction

In macroeconomics, GDP growth is one of the values in which policy planners and economists are most interested. Among other components, consumption is essentially the main element of the GDP in most countries. For example, in the U.S., consumption has been in the high 60 to low 70 percent range for the last ten years. Consumption growth, therefore, highly explains causes of economic growth; and there have been plenty of theoretical and empirical studies of consumption growth, as well as its definition and effects. Examples are Cass (1965), Koopmans (1967), Campbell and Mankiw (1989), and Attanasio and Weber (1993). Generally, economists are interested in consumption level and growth whereas investment planners tend to pay more attention to optimal policy for investment. The paper aims to focus on both issues by starting with the consumption growth problem based on basic models from Munk (2012). Also, with a simple dynamic asset allocation problem, the optimal consumption and investment strategy can be solved for exact solutions and it is easier to further study more in-depths about consumption growth.

Dynamic asset allocation within continuous-time framework has been intensively studied since the seminal work of Merton (1969, 1971). The study helps an investor find continuous-time consumption-investment strategies. The standard method of investigation is based on the stochastic control framework and the Hamilton-Jacobi-Bellman (HJB) equation, resulting in nonlinear equations which are typically difficult to solve. Munk (2012)

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considers a simple case of dynamic asset allocation with constant investment opportunities. For the CRRA utility maximization problem in a market with constant interest rate, expected rate of returns, and volatility, it derives the optimal investment strategy and optimal consumption rate for *d*-asset cases. Using his outlined framework, further research can be done on consumption growth in this research. Obstfeld (1994), among the others, is the first to develop a dynamic asset allocation model and connects the allocation problem with the consumption growth issue. It concludes that international investments can yield welfare gains through its positive effect on expected consumption growth. In fact, the shift from safer but low-yield investment to risky but high-yield helps increase the welfare gains. However, it does not explain the effect of interest rates and other important parameters to the growth.

The concept of Keynes-Ramsey rules (KRR) has been derived since the work of Cass (1965) and Koopmans (1967) in a deterministic growth model. The rule shows how optimal consumption changes over time and provides ideas of factors that determine consumption growth. Recent studies include Turnovsky (2000) who derives the KRR in a model of stochastic growth with Brownian motion, Steger (2005) who shows the rule in an AK-type growth model with Poisson processes, or Sennewald and Wälde (2006) who relates the rule with an optimal consumption problem. We have three crucial differences compared to Sennewald and Wälde (2006). First, we follow the Brownian motion whereas they use Poisson uncertainty as a noise for the stochastic process. Moreover, we allow for *d* risk assets and a risk-free one while Sennewald and Wälde use only one risky asset and a risk-free asset. Finally, whereas it forms the infinite horizon model as an indirect utility function, this current study, instead, forms the finite horizon model. Doing so complicates the problem and leads to the HJB equation in a partial differential equation rather than an ordinary differential equation in the infinite horizon model. Wälde (2011) applies the same method as in his previous work to find the KRR for an intertemporal utility maximization under Brownian motion but still dealing with one asset foundation. Although it is a good start for studying the KRR under the consumption-investment problem, it ignores an analysis of the derived KRR.

Apart from the importance of what defines consumption growth, effects of those elements on consumption need further consideration. The most important factor, perhaps, is the interest rate since it is the most basic investment returns and it can also be determined and controlled from a central bank in each nation. The relationship between consumption growth and the interest rate has been studied both theoretically and empirically. Empirical studies, e.g., Campbell and Mankiw (1989) and Attanasio and Weber (1993), suggest that the interest rate has a positive relationship with consumption growth.

The paper aims at finding a KRR and an expected growth rate of consumption of the simple dynamic consumption and investment problem for a finite model. However, doing so can also provide a result of an infinite time horizon model. A derivation of a HJB equation for an optimal allocation problem is presented in this work. As reviewed earlier that previous studies commonly assume only one risky asset framework with the infinite model which simply can be solved, this current study will compare its results with those previous works. Moreover, this research considers relationships between changes in interest rates and other important parameters with changes in consumption growth. Finally, to relax the assumption of constant interest rates, it also assumes the case with stochastic interest rates and examines what will happen to the KRR and consumption growth.

The rest of the paper is organized as follows. Section 2 sets the model for a simple dynamic consumption and investment problem. Section 3 derives the KRR for this simple dynamic asset allocation case. Section 4 analyzes such a KRR and growth rate and investigates more in other effects. Section 5 provides a numerical example. Finally, Section 6 concludes the paper.

2. The Model

Starting from this section, mathematical vectors and matrices will be used as several financial instruments are to be involved. All vectors are denoted with one underline and matrices with two underlines. The superscript T on a vector or a matrix indicates that the vector or matrix is transposed. The notation $\underline{1}$ is used for a vector where all elements are equal to 1; the dimension of the vector will be clear from the context. Also note that $\|\underline{x}\|^2 = \underline{x}^{\mathrm{T}}\underline{x} = \sum_{i=1}^{n} x_i^2$.

The wealth dynamics, W, for a given consumption strategy c and a given portfolio process can be written as

$$\mathrm{d}W_t = W_t\left[r_t + \underline{\pi}_t^{\mathrm{T}}\underline{\underline{\sigma}}\underline{\lambda}\right]\mathrm{d}t - c_t\mathrm{d}t + W_t\underline{\pi}_t^{\mathrm{T}}\underline{\underline{\sigma}}\mathrm{d}\underline{z}_t. \tag{1}$$

where r_t denotes the interest rate, $\underline{\underline{\sigma}}$ is a volatility matrix of d-assets, $\underline{z}_t$ denotes a d-dimensional standard Brownian motion, and $\underline{\lambda}$ measures the excess rate of return relative to the standard deviation and is given by

$$\underline{\lambda} = \left(\underline{\underline{\sigma}}\right)^{-1}\left(\underline{\mu} - r\underline{1}\right),$$

with $\underline{\mu}$ denoting a drift vector of d assets.

An investor is assumed to maximize utility from the intermediate consumption and from the terminal wealth. The indirect utility function is given by

$$J(W,t) = \sup E_{W,t}\left[\int_t^T \mathrm{e}^{-\delta(s-t)}u(c_s)\mathrm{d}s + \mathrm{e}^{-\delta(T-t)}\overline{u}(W_T)\right]. \tag{2}$$

For a power or constant relative risk aversion (CRRA) utility function which can be written as

$$u(c) = \varepsilon_1\frac{c^{1-\gamma}}{1-\gamma},\ \overline{u}(W) = \varepsilon_2\frac{W_T^{1-\gamma}}{1-\gamma}.$$

Then, from Munk (2012), the indirect utility of wealth function of a CRRA investor is given by

$$J(W,t) = g(t)\frac{W^{1-\gamma}}{1-\gamma}, \tag{3}$$

with

$$g(t)=\frac{1}{A}\left(\varepsilon_1^{1/\gamma}+\left[\varepsilon_2^{1/\gamma}A-\varepsilon_1^{1/\gamma}\right]\mathrm{e}^{-A(T-t)}\right),$$

where

$$A=\frac{\delta+r(\gamma-1)}{\gamma}+\frac{1}{2}\frac{\gamma-1}{\gamma^2}\|\underline{\lambda}\|^2.$$

The optimal investment strategy is given by

$$\underline{\Pi}(W,t)=\frac{1}{\gamma}\left(\underline{\underline{\sigma}}^{\mathrm{T}}\right)^{-1}\underline{\lambda}=\frac{1}{\gamma}\left(\underline{\underline{\sigma\sigma}}^{\mathrm{T}}\right)^{-1}\left(\underline{\mu}-r\underline{1}\right). \tag{4}$$

The optimal consumption rate is

$$c(W,t)=\varepsilon_1^{1/\gamma}\frac{W}{g(t)}=A\left(1+\left[\frac{\varepsilon_2^{1/\gamma}}{\varepsilon_1^{1/\gamma}}A-1\right]\mathrm{e}^{-A(T-t)}\right)^{-1}W. \tag{5}$$

As can be seen from the optimal consumption-investment strategy, the optimal consumption plan is to consume a time-varying fraction of wealth whereas the optimal investment policy does not rely on the investor's time horizon. Moreover the risk-aversion factor has a negative relationship with allocation in risky assets whereas it has a positive relationship with the risk-free asset.

3. The Keynes-Ramsey Rule

The Keynes-Ramsey rule (KRR) is one of the best-known concepts and most widely used in Economics. The KRR characterizes the optimal rate of consumption change over time. In the traditional deterministic model, it says that consumption increases when the interest rate is higher than the time preference rate. More details can be found in Barro and Sala-i-Martin (2003). This section examines the rule in the stochastic framework through a consumption-investment problem with many risky assets. Following the method used in Sennewald and Wälde (2006), the first step of finding the KRR is to write the HJB equation

$$\delta J(W,t)=u(c)-cJ_W+WJ_W\underline{\pi}_t^{\mathrm{T}}\underline{\underline{\sigma}}\underline{\lambda}+J_t+\frac{1}{2}J_{WW}W^2\underline{\pi}_t^{\mathrm{T}}\underline{\underline{\sigma\sigma}}^{\mathrm{T}}\underline{\pi}_t+rWJ_W. \tag{6}$$

The first-order condition for the above maximization is

$$u'(c)=J_W(W,t). \tag{7}$$

Then it is necessary to compute the derivative of the equation with respect to wealth as doing so will provide the shadow price, J_W. Using the envelope theorem, the derivative can be expressed as

$$\begin{aligned}\delta J_W &= u'(c)c(W) + \left(r + \underline{\pi}_t^{\mathrm{T}} \underline{\underline{\sigma}} \underline{\lambda}\right)\left(W J_{WW} + J_W\right) - \left[c J_{WW} + J_W c'(W)\right] \\ &\quad + J_{tW} + \frac{1}{2} \underline{\pi}_t^{\mathrm{T}} \underline{\underline{\sigma\sigma}}^{\mathrm{T}} \underline{\pi}_t \left(W^2 J_{WWW} + 2 J_{WW} W\right).\end{aligned} \tag{8}$$

Now, using Ito's lemma to J_W, the following process can be obtain

$$\mathrm{d}J_W = \left\{ J_{Wt} + J_{WW}\left[W\left(r + \underline{\pi}_t^{\mathrm{T}} \underline{\underline{\sigma}} \underline{\lambda}\right)\right] + \frac{1}{2} J_{WWW} W^2 \underline{\pi}_t^{\mathrm{T}} \underline{\underline{\sigma\sigma}}^{\mathrm{T}} \underline{\pi}_t - C \right\} \mathrm{d}t + J_{WW} W \underline{\pi}_t^{\mathrm{T}} \underline{\underline{\sigma}} \mathrm{d}\underline{z}_t. \tag{9}$$

Thus, rearranging (8) and inserting into (9) gives

$$\mathrm{d}J_W = \left\{\left[\delta - \left(r + \underline{\pi}_t^{\mathrm{T}} \underline{\underline{\sigma}} \underline{\lambda}\right)\right] J_W - J_{WW} W \underline{\pi}_t^{\mathrm{T}} \underline{\underline{\sigma\sigma}}^{\mathrm{T}} \underline{\pi}_t \right\} \mathrm{d}t + J_{WW} W \underline{\pi}_t^{\mathrm{T}} \underline{\underline{\sigma}} \mathrm{d}\underline{z}_t. \tag{10}$$

Replacing, according to the first-order condition (7) for optimal consumption, the following process is obtained

$$\mathrm{d}u'(c) = \left\{\left[\delta - \left(r + \underline{\pi}_t^{\mathrm{T}} \underline{\underline{\sigma}} \underline{\lambda}\right)\right] u'(c) - u''(c) c_W W \underline{\pi}_t^{\mathrm{T}} \underline{\underline{\sigma\sigma}}^{\mathrm{T}} \underline{\pi}_t \right\} \mathrm{d}t + u''(c) c_W W \underline{\pi}_t^{\mathrm{T}} \underline{\underline{\sigma}} \mathrm{d}\underline{z}_t. \tag{11}$$

The above equation describes the evolution of marginal utility from consumption. Nevertheless, it would be more useful to have a Keynes-Ramsey rule for the optimal consumption itself.

Defining $f\left(u'(c)\right) = c$ and applying Ito's lemma to $f\left(u'(c)\right)$, it leads to

$$\begin{aligned}\mathrm{d}f &= \left\{ \begin{matrix} \left(\left[\delta - \left(r + \underline{\pi}_t^{\mathrm{T}} \underline{\underline{\sigma}} \underline{\lambda}\right)\right] u'(c) - u''(c) c_W W \underline{\pi}_t^{\mathrm{T}} \underline{\underline{\sigma\sigma}}^{\mathrm{T}} \underline{\pi}_t \right) f'\left(u'(c)\right) \\ + f_t + \frac{1}{2}\left(u''(c) c_W W\right)^2 \underline{\pi}_t^{\mathrm{T}} \underline{\underline{\sigma\sigma}}^{\mathrm{T}} \underline{\pi}_t f''\left(u'(c)\right) \end{matrix} \right\} \mathrm{d}t \\ &\quad + f'\left(u'(c)\right) u''(c) c_W W \underline{\pi}_t^{\mathrm{T}} \underline{\underline{\sigma}} \mathrm{d}\underline{z}_t.\end{aligned} \tag{12}$$

Multiplying $-u''(c)/u'(c)$ to both sides yields

$$\begin{aligned}-\frac{u''(c)}{u'(c)} \mathrm{d}c &= \left\{ r + \underline{\pi}_t^{\mathrm{T}} \underline{\underline{\sigma}} \underline{\lambda} - \delta + \frac{u''}{u'} c_W W \underline{\pi}_t^{\mathrm{T}} \underline{\underline{\sigma\sigma}}^{\mathrm{T}} \underline{\pi}_t + \frac{1}{2} \frac{u''}{u'} \left(c_W W\right)^2 \underline{\pi}_t^{\mathrm{T}} \underline{\underline{\sigma\sigma}}^{\mathrm{T}} \underline{\pi}_t - \frac{u''}{u'} c_t \right\} \mathrm{d}t \\ &\quad - \frac{u''}{u'} c_W W \underline{\pi}_t^{\mathrm{T}} \underline{\underline{\sigma}} \mathrm{d}\underline{z}_t.\end{aligned} \tag{13}$$

For the CRRA utility function, $-u''(c)/u'(c) = \gamma / c$ and $u'''(c)/u'(c) = \gamma(\gamma+1)/c^2$.

Inserting above relations into (13), the following theorem can be received

$$\begin{aligned}\frac{\mathrm{d}c}{c} &= \frac{1}{\gamma}\left\{ r + \underline{\pi}_t^{\mathrm{T}} \underline{\underline{\sigma}} \underline{\lambda} - \delta - \frac{\gamma}{c} c_W W \underline{\pi}_t^{\mathrm{T}} \underline{\underline{\sigma\sigma}}^{\mathrm{T}} \underline{\pi}_t + \frac{1}{2}\gamma(\gamma+1)\left(\frac{c_W W}{c}\right)^2 \underline{\pi}_t^{\mathrm{T}} \underline{\underline{\sigma\sigma}}^{\mathrm{T}} \underline{\pi}_t + \frac{c_t}{c} \right\} \mathrm{d}t \\ &\quad + \frac{c_W W}{c} \underline{\pi}_t^{\mathrm{T}} \underline{\underline{\sigma}} \mathrm{d}\underline{z}_t.\end{aligned} \tag{14}$$

This is the Keynes-Ramsey rule describing the evolution of consumption under optimal behavior. Inserting the optimal consumption strategy, c, and its derivatives from (5), the following theorem can be obtained

Theorem 3.1: The Keynes-Ramsey rule based on the simple consumption-investment problem is given by

$$\frac{\mathrm{d}c}{c}=\left\{\frac{r-\delta}{\gamma}+\frac{\gamma+1}{2\gamma^2}\|\underline{\lambda}\|^2+\left(g(t)\right)^{-1}\left[\varepsilon_1^{1/\gamma}-\varepsilon_2^{1/\gamma}A\right]\mathrm{e}^{-A(T-t)}\right\}\mathrm{d}t+\frac{1}{\gamma}\underline{\lambda}^{\mathrm{T}}\mathrm{d}\underline{z}_t, \tag{15}$$

Forming expectation yields the average consumption growth rate

$$\frac{\mathrm{E}(\mathrm{d}c)}{\mathrm{d}t\cdot c}=g_c=\frac{r-\delta}{\gamma}+\frac{\gamma+1}{2\gamma^2}\|\underline{\lambda}\|^2+\left(g(t)\right)^{-1}\left[\varepsilon_1^{1/\gamma}-\varepsilon_2^{1/\gamma}A\right]\mathrm{e}^{-A(T-t)}. \tag{16}$$

The term $(r-\delta)/\gamma+\left[(\gamma+1)/2\gamma^2\right]\|\underline{\lambda}\|^2$ is a constant and the term $\left(g(t)\right)^{-1}\left[\varepsilon_1^{1/\gamma}-\varepsilon_2^{1/\gamma}A\right]\mathrm{e}^{-A(T-t)}$ clearly depends on time horizon. Note that in the case of infinite horizon, the indirect utility function is in the form

$$J(t)=\sup E_t\left[\int_t^{\infty}\mathrm{e}^{-\delta(s-t)}u(c_s)\mathrm{d}s\right],$$

where the expected utility consists of the infinite intermediate wealth only with no terminal wealth utility. Solving the maximization problem according to such a utility yields the Keynes-Ramsey rule with the same constant as in (16) but there will be no time horizon related term.

From the Keynes-Ramsey rule (15), the proportional change of consumption is explained by both deterministic and stochastic terms. The former consists of the usual difference between the interest rate and time preference rate plus the "λ-term" which captures the impact of market prices of risk and plus the "$g(t)$-term" which is the effect of introducing time horizon into the problem. To understand the meaning of the whole deterministic term, it is needed to examine the "$g(t)$-term" term. The latter, the "$\mathrm{d}\underline{z}_t$-term", gives changes in consumption in the stochastic way over time too. It can be shown that the value of function g(t) is always non-negative (see Munk, 2012), and the parameter A is a small positive value so it can be roughly implied that the time-varying term is greater than zero. Therefore, as long as prices of risky assets do not change, optimal consumption grows by the positive value of (16).

In the case of infinite horizon, the higher the $\|\underline{\lambda}\|^2$, and the lower the time preference rate, δ, and the risk-aversion factor, γ, the higher becomes the consumption growth rate. However, introducing the finite horizon complicates this deterministic term with the addition of the "$g(t)$-term" term, the whole growth is difficult to interpret due to the time horizon related term. Next section provides more details which lead to better interpretation of the consumption issue.

4. Result Analysis

4.1 The Interest Rate Effect on Consumption

From the expected growth rate of consumption (16), it involves three terms as explained earlier in the last section. For the infinite horizon model, the interest rate affects the growth only via constant terms, the traditional KRR and the "λ-term" terms. However, movements in the interest rate not only positively affect consumption growth, but also indirectly and negatively affect the growth via the market prices of risk, $\underline{\lambda} = (\underline{\underline{\sigma}})^{-1}(\underline{\mu} - r\underline{1})$. Such a result is different from the result of Sennewald and Wälde (2006) and other traditional works stating that the higher risk-free rate leads to higher consumption growth rate. This is mainly due to an introduction of risky assets. For simplicity, if there only exists one risky asset in the infinite horizon model, the derivative of the average growth rate with respect to the interest rate is

$$\frac{\partial g_c}{\partial r} = \frac{1}{\gamma}\left[1 - \left(1 + \frac{1}{\gamma}\right)\left(\frac{\mu - r}{\sigma^2}\right)\right], \tag{17}$$

where μ is the mean returns and σ^2 is the variance of the asset. From the equation, changes in the interest rate will have a positive relationship with the average growth rate when $(1 + 1/\gamma)(\mu - r)/\sigma^2 < 1$. The main point here is that not only does an increase in the interest rate positively affects the consumption growth via cash investment, but it also negatively affects the expected returns in excess of the risk-free rate via risky assets allocation. Moreover, when investors are especially less afraid of risks, the effect of interest rate is even higher.

When considering the finite horizon model, there exists the time horizon related *g(t)* term which confuses investigation of the interest rate effect on the average growth. Further numerical examples, therefore, are required to help present the effect of the time horizon along with complementing the above analysis.

4.2 Other Effects on Consumption

Time horizon effect

As shown in (16), expected consumption growth of the infinite horizon model is independent from the investment horizon. Thus, longer or narrower consumption-investment time horizon will not affect the average consumption growth. However, for the finite horizon model, it is clear that investment horizon negatively affects expected consumption growth. The following equation computes the derivative of average growth with respect to the time horizon

$$\frac{\partial g_c}{\partial \tau} = -A^2\left[1 + \frac{e^{-A(T-t)}}{\left(1 - e^{-A(T-t)}\right)^2}\right]. \tag{18}$$

The inverse relation is due to the fact that the average growth is actually the percentage change of consumption divided by time horizon. Hence, when the time horizon is longer, the total value of the average growth is lower.

Risk aversion effect

The derivative of the average growth with respect to the risk-aversion factor can be written as

$$\frac{\partial g_c}{\partial \gamma} = -\left[\frac{r-\delta}{\gamma} + \frac{\gamma^2 + 2\gamma}{2\gamma^4}\|\underline{\lambda}\|^2\right] + \frac{\partial}{\partial \gamma}\left(g(t)\right)^{-1}\left[\varepsilon_1^{1/\gamma} - \varepsilon_2^{1/\gamma} A\right]\mathrm{e}^{-A(T-t)}. \tag{19}$$

From the equation, given that the interest rate is higher than the time preference rate, the above equation shows that expected consumption growth of the infinite horizon model has a negative relationship with the risk-aversion factor. The result agrees with previous studies such as Sennewald and Wälde (2006) and others. From such consequence, the inverse of the risk-aversion coefficient, γ^{-1}, is, thus, called the intertemporal elasticity of consumption substitution. The more willing households are to shift consumption over time (γ^{-1} is high), the steeper the consumption path will be. However, combining the time related term complicates the whole term so that it is difficult to state how changes in the risk-aversion factor affects the growth rate.

In fact, the consumption growth (16) can be rewritten as

$$g_c = \frac{1}{\gamma}\left(r-\delta\right) + \left(\frac{1}{2\gamma} + \frac{1}{2\gamma^2}\right)\|\underline{\lambda}\|^2 + \left(g(t)\right)^{-1}\left[\varepsilon_1^{1/\gamma} - \varepsilon_2^{1/\gamma} A\right]\mathrm{e}^{-A(T-t)}. \tag{20}$$

The value $(1/2\gamma)+(1/2\gamma^2)$, can be regarded as the coefficient of market prices of risk, $\|\underline{\lambda}\|^2$. When γ is equal to 1 and 10, this coefficient will equal 1 and 0.055, respectively, a huge difference of coefficients. When the coefficient is very low, changes in interest rate will barely affect consumption growth. Besides, the market prices of risk are very sensitive to risk-aversion factor variations.

5. A Numerical Example

Three main questions to be answered are (1) how investors allocate their portions in assets; (2) how infinite consumption growth differs from finite growth; and (3) how changes in interest rates affect consumption growth. To respond to those points, this section illustrates the simple consumption-investment strategy with 6 risky assets in Theorem 6.2 of Munk (2012) and average consumption growth with historical estimates of international stock returns, standard deviations, and correlations as representative of future investment opportunities. Estimates are taken from Munk (2012) and Das and Uppal (2004). The value of the equity index consists of the U.S. (USA), Argentina (ARG), Hong Kong (HKG), Mexico (MEX), Singapore (SNG), and Thailand (THA). Table 1 reports the descriptive statistics taken from [III] where the returns are in the U.S. dollar value of the index. The average real U.S. short-term interest rate is 1.00%.

Table 1. Returns, volatilities, and correlations from Das and Uppal (2004)

1-year	USA	ARG	HKG	MEX	SNG	THA
Returns	12.48%	4.80%	9.12%	4.08%	7.80%	0.05%
Volatility	14.34%	74.58%	35.54%	49.78%	26.74%	35.92%

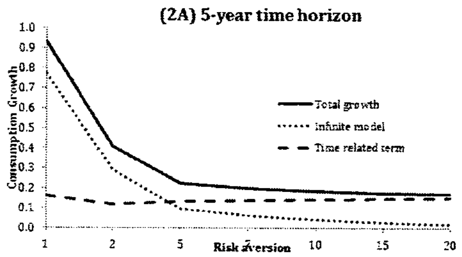
Correl	ARG	HKG	MEX	SNG	THA
USA	0.1039	0.4051	0.3586	0.5519	0.3445
ARG		0.058	0.2167	0.0842	0.1286
HKG			0.2475	0.5479	0.4347
MEX				0.3543	0.2972
SNG					0.5291

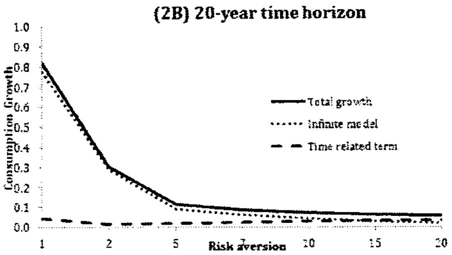
5.1 Infinite and Finite Consumption Growth

Plugging estimates into (20) provides consumption growth at a determined risk-aversion factor, an investment time horizon, non-negative coefficients ε_1 and ε_2 (assuming 1 and 1). Figure 2 illustrates the relation between consumption growth (both infinite and finite growth) and risk aversion with 5-year and 20-year investment time horizons, respectively. As mentioned earlier, finite consumption growth consists of three terms, the traditional KRR $(r-\delta)/\gamma$, the market prices of risk term, and the time-varying term. The first two terms are constant while the last is not. For the infinite case, the growth is an addition of the traditional KRR and the market prices of risk term while for the finite case, the growth is a combination of the infinite consumption growth and the time horizon related term. The value of the traditional KRR is very low due to lower the numerator of lower than 0.01 and the denominator of not less than the unity. As can be seen from the figure, both infinite and finite consumption growths and risk-aversion factors have an inverse relationship, implying that when investors are more risk-averse, they tend to consume more in the present rather than in the future. Compared to the impact of the constant term or the infinite consumption growth, the time related term play a minor role in determining the growth for the finite case when investors have low risk-averse.

Figure 2 also points out impacts of different time horizons. As can be seen in figure 2A. , for shorter time horizon framework, the time related term has more impacts when investors have higher risk-averse but less impact in the opposite. To summarize, the longer the time horizon, the closer of infinite and finite consumption growth and the lower effect of the time horizon related term. Therefore, when considering a very long time horizon of consumption and investment, the time horizon related term can simply be ignored. On the contrary, when considering a shorter time framework with high risk-averse investors, it is important to include the effect of time horizon related term otherwise the analysis will significantly be wrong.

Figure 2. Relation between consumption growth (infinite and finite growth) and risk aversion





The dashed lines correspond to infinite consumption growth, the dotted line corresponds to the time horizon related term, and the solid lines correspond to finite consumption growth.

5.2 Various effects on Consumption Growth

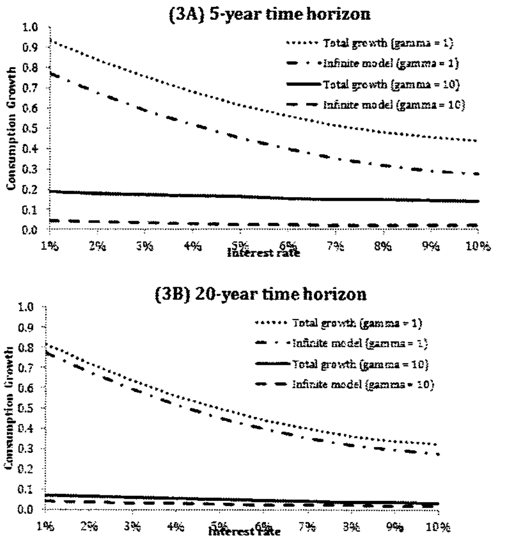
Figure 3 shows the relation between interest rate changes and consumption growth with 5-year and 20-year investment time horizons, respectively. As can be seen, interest rates have a negative relationship with both infinite and finite consumption growth and the effect is especially high when investors are less risk-averse (see dotted and dot-dashed lines of both figures). That is to say, investors with low risk aversion coefficient are more susceptible to changes in interest rates. For more clarification, when investors are less afraid of risks, they tend to allocate much more money in risky assets. Movements in the interest rate, thus, vastly shake their returns excess of risk-free rate or market prices of risk. The result implies that when the whole vector of equity risk premium is lower (when the interest rate moves upward), investors lose their ability to gain more from risky assets and finally lead to a decrease in consumption growth. Conversely, when investors are relatively more afraid of risks as in the case of solid and dashed lines for finite and infinite horizon models, respectively, changes in interest rates rarely affect consumption growth. In fact, we can rewrite the consumption growth (20) into

$$g_c = \frac{1}{\gamma}(r-\delta) + \left(\frac{1}{2\gamma} + \frac{1}{2\gamma^2}\right)\|\underline{\lambda}\|^2 + (g(t))^{-1}\left[\varepsilon_1^{1/\gamma} - \varepsilon_2^{1/\gamma} A\right]e^{-A(T-t)}. \tag{21}$$

Considering the first term of (21), the traditional KRR, the effect of risk aversion coefficient on interest rates is called the elasticity of intertemporal substitution (or just the inverse of risk aversion coefficient). The advent of the "λ-term" in the study is very important that the coefficient of the square of market prices of risk vector, or $(1/2\gamma)+(1/2\gamma^2)$, is significantly less than the inverse of risk-aversion factor. For example, when γ is equal to 1 and 10, this coefficient will equal 1 and 0.055, respectively, a huge difference of coefficients. When the coefficient is very low, changes in interest rate barely affect consumption growth. In sum, the market prices of risk term are very sensitive to risk-aversion factor variations.

Considering infinite and finite growth, the finite growth is only the parallel shift of the infinite growth as we can see from the shift of dot-dashed lines to dotted lines and dashed lines to solid lines. Also, the longer the time horizon, the closer of infinite and finite consumption growth and the lower effect of the time horizon related term. As we can see that gaps between lines are closer in 20-year time horizon model. This is no surprise because when time horizon approaches infinity, the finite model lastly becomes the infinite model.

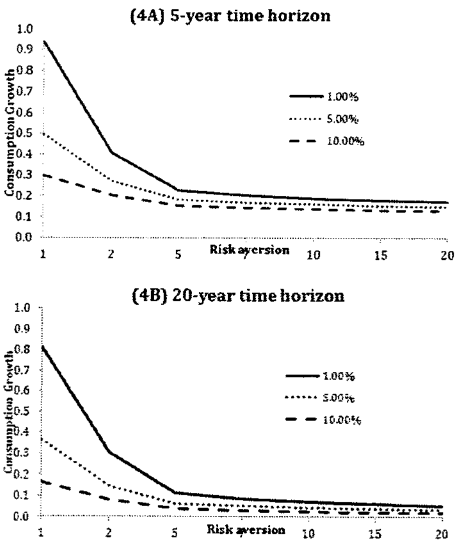
Figure 1. Interest rate effect on consumption growth



The dotted lines correspond to the growth rate of finite model with risk aversion coefficient of 1, the dot-dashed lines correspond to the growth rate of infinite model with risk aversion coefficient of 1, the solid lines correspond to the growth rate of finite model with risk aversion coefficient of 10, and the dashed lines correspond to the growth rate of infinite model with risk aversion coefficient of 10.

Figure 4 shows the relation between consumption growth and risk aversion with 5-year and 20-year investment time horizons, respectively. Consumption growth has a negative relationship with risk-aversion factor. This implies that when investors are more risk-averse, they tend to consume more in the present rather than in the future. From the figure, the effect of interest rates can also be seen clearer. Interest rate movements negatively affect the growth of consumption, same as explained earlier in the previous subsection. As can be seen from both figures, for investors with risk-aversion factor of lower than 5, consumption growth among different risk-free rates are high. Also, for longer consumption-investment time horizon, the growth is lower.

Figure 2. Effects on consumption growth



Relation between consumption growth and risk aversion. The solid lines correspond to the growth with risk-free asset of 1.00%, the dashed lines correspond to the growth with risk-free asset of 5.00%, and the dotted lines correspond to the growth with risk-free asset of 10.00%.

6. Summary and Conclusions

This study computed the Keynes-Ramsey rule (KRR) and an expected consumption growth rate for a simple dynamic asset allocation problem using the HJB equation for infinite and finite models. The paper contributes to the field by answering important economic questions including (1) what explains consumption growth and (2) how certain internal and external factors affect them.

Regarding to results from Section III, deriving an expected growth rate of consumption provides different results for infinite and finite models. For the infinite horizon model, consumption growth constitutes the conditional difference between the interest rate and time preference rate over risk aversion coefficient; and the term capturing the impact of market prices of risk. The finite model growth; however, is a combination of the infinite model growth and the time-varying term, which is the effect of introducing time horizon into the problem.

Moreover, interest rate movements not only positively affect consumption growth, but also indirectly and negatively affect the growth via the market prices of risk. This study also uncovered an inverse relationship between time horizon and the consumption growth rate. In addition, the result recommends a negative relationship between risk-aversion factor and consumption growth, implying that a more risk-averse individual is more-contented with current consumption. However, due to the existence of the time-varying term, there are some unsettled parts that need to be confirmed and illustrated numerically. This could be a topic of further studies.

The numerical example results in two noteworthy consequences. The time-varying term in the finite model plays an important role to consumption growth when consumption-investment time horizon is short and investors are more afraid of risks. Moreover, changes in interest rates have a negative relationship with both infinite and finite consumption growth; and the effect is especially large if investors are less risk-averse.

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